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Separation from a smooth surface in a slender conical flow

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SUMMARY

High Reynolds number flow of an incompressible fluid past a smooth surface in a slender conical flow is considered. Attention is focused upon the flow properties in the neighbourhood of the separation line. The analysis incorporates the results of a recent inviscid-flow investigation by Smith [1], and the ideas of Sychev [2] for flow separation in two dimensions.

1. Introduction

In this paper we consider the nature of the high Reynolds number flow in the neighbourhood of the separation line on a smooth surface in a slender conical flow. The analysis is based upon an inviscid model of the flow proposed by Smith [1], and incorporates the ideas developed by Sychev [2] for the high Reynolds number flow close to a separation point in two dimensions.

The separation phenomenon in a conical flow differs from that in two dimensions in that for the latter a closed separation bubble is formed when the flow is steady in which the total head of the fluid is very much less than in the stream outside. The 'open' type of separation associated with conical flow is also a much more stable phenomenon than the closed-bubble separation of two dimensions.

A frequently encountered type of separation in conical flow is the separation from a salient edge, as in leading-edge separation. In such a flow the separation line is pre-determined, and for an inviscid fluid the separating flow may be modelled using an infinite spiral vortex sheet springing from the edge. Successful numerical calculations have been performed, originally by Smith [3], using such a model in which the vortex sheet forms a stream surface of the flow and across which there is no jump in pressure; in addition a Kutta condition is applied at the edge. Separation in a conical flow can also take place away from salient edges, for example primary separation on a circular cone at incidence or the secondary separation beneath the primary vortex on a delta wing. In each of these cases the separation takes place from a smooth surface and, in view of the success experienced with the leading-edge vortex flow, it is natural to enquire whether or not a theory for separation from a smooth surface can be developed from an inviscid model based upon an infinite spiral vortex sheet originating at the separation line. For such a model the separation line must be fixed by viscous considerations. Nutter [4] has enjoyed partial success in his calculations of the secondary separation from the surface of a slender delta wing. In these calculations the secondary separation is fixed and the inviscid-flow characteristics evaluated; the boundary layer on the wing surface is then investigated numerically. The position of the separation line for the inviscid flow is systematically changed until it coincides with the separation line predicted by the boundary-layer calculation. Nutter observes from his calculations that separation of the flow takes place just beyond the pressure minimum following a small but sharp rise in pressure. However it should be noted that there is only poor numerical resolution close to the separation line in both the inviscid and viscous parts of these calculations.

It is clear that for a further understanding of these separated flows a more detailed knowledge of the flow field close to the separation line is required. Smith [1] has recently carried out a detailed inviscid analysis in the neighbourhood of the separation line at which a vortex sheet leaves the smooth surface. He finds a solution in which the vortex sheet has infinite curvature and in which the pressure gradient becomes infinite as separation is approached. In this respect the flow properties are similar to those in the classical two-dimensional Kirchhoff model of separated flow. Smith also discusses another solution which does not exhibit this singular behaviour, and which is analogous to inviscid smooth separation in two dimensions (see for example Thwaites [5]).

The inviscid solutions of Smith provide the starting point for the high Reynolds number investigation of this paper in which we consider the slender conical flow along a plane wall with flow separation taking place along a conical ray. It is first shown that the singular inviscid solution cannot be consistent with classical boundary-layer theory. Following Sychev [2] we then postulate, as the basis for a self-consistent theory, that the singular solution incorporates a multiplicative constant $O(R^{-\frac{1}{16}})$ where $R = U\ell/\nu$ is the Reynolds number. Here U is the free stream speed, ℓ a length and ν the kinematic viscosity of the fluid. We assume $R \gg 1$, and develop a theory appropriate in the formal limit $R \to \infty$. In the inviscid limit $R = \infty$ the singular solution referred to above does not feature.

Because the changes which take place normal to the separation line in this slender conical flow are very rapid, compared with those parallel to it, the flow structure exhibits many of the features of a two-dimensional flow. Sychev [2] has considered the two-dimensional case and our development closely parallels his work. Thus we identify three distinct regions in the flow, namely the pre-interaction region, the interaction region itself in which separation takes place and the post-interaction region. In the first of these regions the classical boundary layer develops in a pressure gradient which becomes very large. The boundary layer then exhibits a double structure as the inner parts respond to the rapid changes in pressure. This region matches with the interaction region in which the solution exhibits a triple-deck structure (see for example Stewartson [6]), and in which the two-dimensional calculations of Smith [7], predicting separation, can be incorporated. The final region downstream from the interaction region has itself to be divided into three parts. There is a boundary layer at the wall, a shear layer centred upon the vortex sheet and a further adjustment region in order that a satisfactory match with the triple-deck can be made. The flow in this post-interaction region is more complicated than in two dimensions. This is because the 'dead-air' region of two dimensions is replaced by a region in which the fluid velocity is the same order of magnitude as that upstream of separation.

We note finally that the inviscid ideas of [1] have been extended by Smith [8] to three-dimensional flows. He also shows that the triple-deck structure is appropriate close to the separation line. Although, in a sense, the slender conical flow is a special three-dimensional flow it is worthy of a discussion in its own right because the inviscid flow beyond separation is locally determined. This is not so in a general three-dimensional flow. Thus a more complete theory can be developed for the case of the slender conical flow.

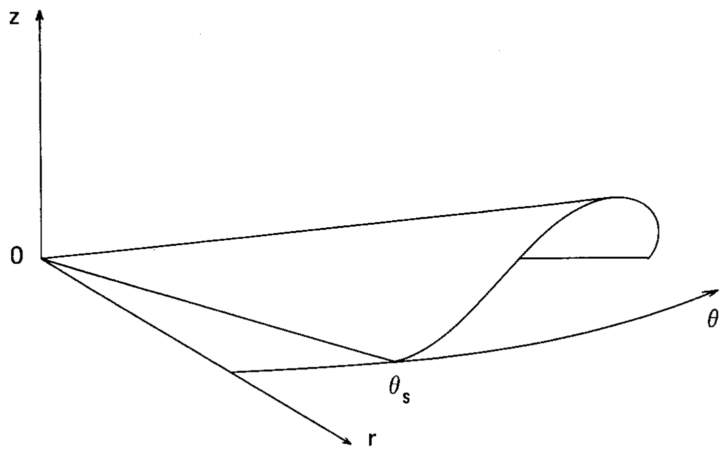

Figure 1. The inviscid separating vortex sheet and co-ordinate system.

2. The inviscid solution

With reference to the cylindrical polar co-ordinate system of Fig. 1 the body shape considered is the plane wall $z = 0$ which extends laterally to infinity, and is parallel to the undisturbed flow which is a uniform stream with speed U in the direction $\theta = 0$. The flow is assumed to be conical, and only highly swept separation lines are considered so that slender-body theory is applicable. Smith [1] has considered the 'open' type of separation, in the inviscid limit $R = \infty$, in which the separated flow is modelled by a slender conical vortex sheet which springs from the separation line $\theta = \theta_s$. In particular, close to the separation line itself, Smith assumes that the shape of the vortex sheet in any surface r = constant is given by

$$z \sim \mu(\theta - \theta_s)^n, \tag{2.1}$$

where n is a parameter to be determined. If $n = \frac{1}{2}(2M + 3)$ he demonstrates that the vortex sheet boundary conditions are satisfied, that is there is no jump in pressure across the sheet which itself forms part of a stream surface, if $M = 0$ or 1 corresponding to $n = \frac{3}{2}, \frac{5}{2}$ respectively. He further conjectures that these conditions will satisfied for the complete set of values of n given by $M = 0, 1, 2 \ldots$.

If we let $\xi = \theta - \theta_s$ then we refer to $\xi < 0$ as the upstream side of the separation line and $\xi > 0$ as the downstream side. Smith shows that the surface velocities and pressure gradient upstream of separation in this conical flow, made dimensionless with U and ρU^2 respectively where ρ is the fluid density, are given for $M = 0$ and $|\xi| \ll 1$ by

$$V_r = 1 + O(\xi),$$
$$V_\xi = c_0 + c_1(-\xi)^{\frac{1}{2}} + O(\xi), \qquad (2.2)$$
$$\frac{\partial p}{\partial \xi} = \tfrac{1}{2}\, c_0 c_1 (-\xi)^{-\frac{1}{2}} + O(1),$$

where V_r, V_ξ are the r- and ξ-components of velocity respectively, and c_0, c_1 are constants. The singular behaviour of these quantities is analogous to the behaviour close to separation in the Kirchhoff free-streamline flow in two dimensions. Smith observes that for $M = 1$ the flow variables exhibit no such singular behaviour. Downstream of separation the flow shows no singular behaviour regardless of the value of M. Thus for $0 < \xi \ll 1$

$$V_r = 1 + O(\xi^2),$$
$$V_\xi = -\frac{4}{2M+5}\xi + O(\xi^2), \qquad (2.3)$$
$$\frac{\partial p}{\partial \xi} = \frac{4(2M+1)}{(2M+5)^2}\xi + O(\xi^2).$$

In the remaining sections of this paper we shall examine the consequences of the predictions of inviscid theory, outlined above, for high Reynolds number flow. In particular we shall argue that the singular solution corresponding to $M = 0$ is suppressed in the formal limit $R \to \infty$.

3. Boundary-layer analysis

With the inviscid flow determined, as described in Section 2, it is important to ensure that this is consistent with a high Reynolds number viscous-flow analysis. In particular viscous-flow separation must take place at $\theta = \theta_s$ in Fig. 1. It is natural therefore to carry out a classical boundary-layer analysis and it proves convenient to employ the reduction of the boundary-layer equations used by Brown [9] for conical flow. Thus we first introduce a dimensionless co-ordinate normal to the boundary as

$$\zeta = R^{\frac{1}{2}} z/(r\ell)^{\frac{1}{2}}, \qquad (3.1)$$

and if (V_r, V_ξ, V_z) are the dimensionless components of velocity we define

$$\overline{V}_z = R^{\frac{1}{2}} V_z (r/\ell)^{\frac{1}{2}}. \qquad (3.2)$$

Following the introduction of these variables all flow properties are independent of r in our conical flow, and if we write

$$V_r = \frac{\partial s}{\partial \zeta}, \quad V_\xi = \frac{\partial \psi}{\partial \zeta},$$
$$\overline{V}_z = \tfrac{1}{2}\,\zeta\frac{\partial s}{\partial \zeta} - \tfrac{3}{2}\,s - \frac{\partial \psi}{\partial \xi}, \tag{3.3}$$

then the boundary-layer equations may be written as

$$-\left(\frac{3}{2}s + \frac{\partial \psi}{\partial \xi}\right)\frac{\partial^2 s}{\partial \zeta^2} + \frac{\partial \psi}{\partial \zeta}\,\frac{\partial^2 s}{\partial \xi \partial \zeta} - \left(\frac{\partial \psi}{\partial \zeta}\right)^2 = \frac{\partial^3 s}{\partial \zeta^3},$$
$$-\left(\frac{3}{2}s + \frac{\partial \psi}{\partial \xi}\right)\frac{\partial^2 \psi}{\partial \zeta^2} + \frac{\partial \psi}{\partial \zeta}\,\frac{\partial^2 \psi}{\partial \xi \partial \zeta} + \frac{\partial \psi}{\partial \zeta}\,\frac{\partial s}{\partial \zeta} = -\frac{dp}{d\xi} + \frac{\partial^3 \psi}{\partial \zeta^3}. \tag{3.4}$$

With the pressure gradient as in (2.2) for the case $M = 0$ we now carry out a local analysis, close to separation, in the spirit of Goldstein [10], Stewartson [11] and Brown [9]. In order to retain a balance between pressure, viscous and inertia terms in (3.4) we write

$$\lambda = (-\xi)^{\frac{3}{8}}, \quad \eta = (c_0 c_1)^{\frac{1}{4}}\zeta/\lambda,$$

with

$$\psi \sim (c_0 c_1)^{\frac{1}{4}}\lambda^{\frac{5}{3}}\bar{f}_0(\eta), \quad s \sim (c_0 c_1)^{\frac{1}{4}}\bar{g}_0(\eta) \quad \text{as } \lambda \to 0. \tag{3.5}$$

The functions $\bar{f}_0$, $\bar{g}_0$ then satisfy

$$\bar{f}_0''' - \tfrac{5}{8}\bar{f}_0\bar{f}_0'' + \tfrac{1}{4}\bar{f}_0'^{\,2} = \tfrac{1}{2},$$

$$g_0''' - \tfrac{5}{8}\bar{f}_0\bar{g}_0'' - \tfrac{3}{8}\bar{f}_0'\bar{g}_0' = 0,$$

with $\bar{f}_0(0) = \bar{f}_0'(0) = \bar{g}_0(0) = \bar{g}_0'(0) = 0$ together with $\bar{f}_0''(0) = 0$, since the transverse skin friction vanishes on the line $\xi = 0$, and the condition that $\bar{f}_0$, $\bar{g}_0$ are not exponentially large as $\eta \to \infty$. From the equation for $\bar{f}_0$ we see that $\bar{f}_0''$ has the properties that

$$\bar{f}_0'' \sim a\eta^{-\frac{1}{3}} \quad \text{as } \eta \to \infty, \ a \text{ const},$$
$$\bar{f}_0'' = \tfrac{1}{2}\,\eta + O(\eta^5), \ \eta \ll 1. \tag{3.6}$$

Equations (3.6) imply that $\bar{f}_0'''$ vanishes for some value of η and we suppose that it vanishes for the first time at $\eta = \eta_0$ where we require $\bar{f}_0^{iv}(\eta_0) \leqslant 0$. Differentiating the equation for f_0 once with respect to η gives

$$\bar{f}_0^{iv} = \tfrac{5}{8}\,\bar{f}_0\bar{f}_0''' + \tfrac{1}{8}\,\bar{f}_0'\bar{f}_0'', \tag{3.7}$$

and since $\bar{f}_0(\eta_0)$, $\bar{f}_0'(\eta_0)$, $\bar{f}_0''(\eta_0) > 0$, $\bar{f}_0'''(\eta_0) = 0$ we deduce that $\bar{f}_0^{iv}(\eta_0) > 0$ which gives a contradiction. On the basis of this analysis we deduce that the local high Reynolds number flow is not consistent with the inviscid flow described in Section 2. We note that for $c_1 = O(1)$ no matter how small, the pressure gradient of (2.2) is infinite at $\xi = 0$ and we might intuitively expect the viscous flow to separate at some $\xi < 0$. In the next section we show how the ideas introduced by Sychev [2] for two-dimensional flow can be employed in our conical flow situation to describe the viscous-inviscid interaction which takes place in the neighbourhood of the separation line.

4. Interaction theory

The situation in Sections 2 and 3 is analogous to that for two-dimensional flow in which the Kirchoff free-streamline theory describes the inviscid separated flow. Sychev [2] has developed a self-consistent analysis for two-dimensional flow in which the coefficient of the singular term in the pressure gradient is not independent of Reynolds number but is vanishingly small as $R \to \infty$. We adopt the approach of Sychev here and assume that the constants μ, c_1 in (2.1), (2.2) are both $O(R^{-\frac{1}{16}})$, specifically we write $c_1 = R^{-\frac{1}{16}} \bar{c}_1$. As in the case of two-dimensional flow this leads to a self-consistent theory as set out below.

The interaction analysis is centred upon the triple-deck theory introduced independently by Messiter [12] and Stewartson [13]. The triple-deck is preceded upstream by a pre-interaction region, which itself exhibits a double structure, and is followed by a post-interaction region which must be studied in several different parts. Figure 2 shows a schematic representation of the situation. The post-interaction region differs in a conical flow from that in a two-dimensional flow. In the latter case, as we proceed beyond separation, we move into what is essentially a 'dead-air' region. For the conical flow under consideration here the separation is of open type and so beyond separation the flow velocities are $O(1)$ in contrast to the much smaller velocities anticipated in a closed separation bubble.

We now discuss the pre-interaction, triple-deck and post-interaction regions separately.

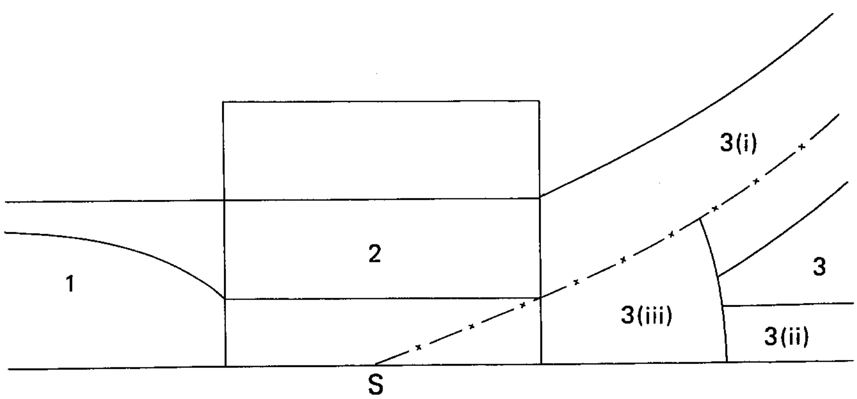

Figure 2. The flow region under consideration. 1 Pre-interaction region, 2 Triple-deck region, 3 Post-interaction region. The post-interaction region itself consists of 3(i) the shear layer, 3(ii) the boundary layer and 3(iii) the adjustment region. S denotes the separation point and the separating vortex sheet of the inviscid flow is denoted thus — x — x — x —.

4.1. Pre-interaction region

With $\partial p/\partial \xi$ from (2.2), given by

$$\frac{\partial p}{\partial \xi} = \alpha_0 + \tfrac{1}{2} R^{-\frac{1}{16}} c_0 \bar{c}_1 (-\xi)^{-\frac{1}{2}} + O(R^{-\frac{1}{8}}), \tag{4.1.1}$$

we have a situation in which, as separation is approached, the boundary layer is developing in a prescribed pressure gradient α_0. For the boundary-layer solution we write

$$\begin{aligned} \psi &= \psi_0(\xi,\zeta) + R^{-\frac{1}{16}} \psi_1(\xi,\zeta) + O(R^{-\frac{1}{8}}), \\ s &= s_0(\xi,\zeta) + R^{-\frac{1}{16}} s_1(\xi,\zeta) + O(R^{-\frac{1}{8}}). \end{aligned} \tag{4.1.2}$$

The leading terms ψ_0, s_0 in (4.1.2) may be expanded as series close to $\xi = \zeta = 0$ in the form

$$\begin{aligned} \psi_0 &= a_0\zeta^2 + a_1\zeta^3 + a_3\zeta^5 + \ldots\ldots + \xi(b_0\zeta^2 + \ldots\ldots), \\ s_0 &= A_0\zeta^2 + A_3\zeta^5 + \ldots\ldots + \xi(B_0\zeta^2 + \ldots\ldots), \end{aligned} \tag{4.1.3}$$

where, from equations (3.4), the unknown constants are determined in terms of A_0, a_0 and α_0 as

$$\begin{aligned} a_1 &= \frac{1}{6}\alpha_0, \quad b_o = \frac{60a_3 - A_0a_0}{2a_0}, \\ B_0 &= \frac{30a_0A_3 + 2a_0^3 + 30a_0A_3 + a_0A_0^2}{a_0^2}, \end{aligned} \tag{4.1.4}$$

and so on.

Now, the second term of (4.1.1) which corresponds to a very rapid change in pressure as $|\xi| \to 0$ will bring about modifications to the leading term in the solution. In particular, as we might suppose, the inner region of the boundary layer develops its own structure in response to this rapid pressure variation. Thus in a region which is of thickness $O(|\xi|^{\frac{1}{3}})$ as $\xi \to 0-$ we write

$$\begin{aligned} \psi_1(\xi,\zeta) &= \frac{c_0\bar{c}_1}{2a_0} (-\xi)^{\frac{1}{2}} f(\eta), \\ s_1(\xi,\zeta) &= \frac{A_0c_0\bar{c}_1}{2a_0^2} (-\xi)^{\frac{1}{2}} g(\eta), \end{aligned} \tag{4.1.5}$$

where

$$\eta = a_0^{\frac{1}{3}} \zeta (-\xi)^{-\frac{1}{3}}.$$

From (3.4), (4.1.3) and (4.1.5) we see that f, g satisfy

$$\begin{aligned} f''' - \tfrac{2}{3}\eta^2 f'' + \eta f' - f &= 1, \\ g''' - \tfrac{2}{3}\eta^2 g'' + \tfrac{1}{3}\eta g' &= f - \tfrac{2}{3}\eta f'. \end{aligned} \tag{4.1.6}$$

The only property of the solution of these equations which we shall require is that as $\eta \to \infty$

$$f \sim \alpha \eta^{\frac{3}{2}} + \alpha_2 \eta, \qquad g \sim \beta \eta^{\frac{3}{2}} + \alpha_2 \eta, \tag{4.1.7}$$

where α, β and α_2 are constants. These asymptotic forms provide the matching condition required for the solution in the outer part of the boundary layer. We do not investigate this outer solution further but we shall incorporate (4.1.7) into the matching associated with the triple-deck.

We note that in this inner region the leading terms of each of ψ_0, s_0 are $O((-\xi)^{\frac{2}{3}}\eta^2)$ and so from (4.1.2) and (4.1.5) we see that the perturbations in the expansions (4.1.2) in this lower region are comparable with the leadings terms when $|\xi| = O(R^{-\frac{3}{8}})$; this is the natural emergence of the triple-deck transverse length scale. Furthermore when $|\xi| = O(R^{-\frac{3}{8}})$ we see from $(4.1.5)_3$ that in this lower region, where $\eta = O(1)$ we have $\zeta = O(R^{-\frac{1}{8}})$. This is the scale normal to the wall of the lower deck in the triple-deck and shows how (4.1.5) merges into the lower deck of the triple-deck structure.

In Section 4.2 we consider the triple-deck and we note that because of the rapid variation in pressure which is predicted close to the separation line, and normal to it, the triple-deck is quasi two-dimensional. It is therefore possible to base the analysis on the two-dimensional theories of Messiter [12], Stewartson [6], [13] and Sychev [2].

4.2. *The triple-deck region*

This central interaction region, as we have already noted, has transverse length scale $O(R^{-\frac{3}{8}})$. The analysis incorporates the triple-deck structure and we now examine in turn the main, lower and upper decks shown schematically in Fig. 2. We note that the essence of the interaction is the coupling between the upper and lower decks in which the solution cannot be determined independently.

(i) *The main deck*

The main deck plays a relatively passive role in the interaction and represents a continuation of the oncoming boundary layer. Thus in the main deck $\zeta = O(1), \xi = O(R^{-\frac{3}{8}})$ and we introduce a new variable

$$\xi^* = R^{-\frac{3}{8}}\xi. \tag{4.2.1}$$

We expand the solution in the main deck in powers of $R^{-\frac{1}{16}}$ as

$$\psi = \psi_{00}(\zeta) + R^{-\frac{1}{16}}\psi_{10}(\zeta) + R^{-\frac{1}{8}}\psi_2(\xi^*, \zeta) + O(R^{-\frac{3}{16}}), \qquad s = s_{00}(\zeta) + R^{-\frac{1}{16}}s_{10}(\zeta) + R^{-\frac{1}{8}}s_2(\xi^*, \zeta) + O(R^{-\frac{1}{16}}), \tag{4.2.2}$$

where $\psi_{00}, \psi_{10}, s_{00}, s_{10}$ are the leading terms, with respect to the variable ξ, in (4.1.2). We note from (4.1.3) and (4.1.7) that as $\zeta \to 0$

$$\psi_{00} \sim a_0\zeta^2, \quad s_{00} \sim A_0\zeta^2,$$
$$\psi_{10} \sim \frac{c_0 c_1 \alpha}{2a_0^{\frac{1}{2}}}\zeta^{\frac{3}{2}}, \quad s_{10} \sim \frac{A_0 c_0 \bar{c}_1 \beta}{2a_0^{\frac{3}{2}}}\zeta^{\frac{3}{2}}. \tag{4.2.3}$$

The equations satisfied by ψ_2, s_2 are the inviscid equations

$$\frac{\partial\psi_{00}}{\partial\zeta}\frac{\partial^2\psi_2}{\partial\xi^*\partial\zeta} - \frac{\partial\psi_2}{\partial\xi^*}\frac{\partial^2\psi_{00}}{\partial\zeta^2} = 0,$$
$$\frac{\partial\psi_{00}}{\partial\zeta}\frac{\partial^2\psi_2}{\partial\xi^*\partial\zeta} - \frac{\partial\psi_2}{\partial\xi^*}\frac{\partial^2 s_{00}}{\partial\zeta^2} = 0, \tag{4.2.4}$$

and from these equations we may deduce that

$$\psi_2 = A(\xi^*)\frac{\partial\psi_{00}}{\partial\zeta}, \quad s_2 = A(\xi^*)\frac{\partial s_{00}}{\partial\zeta}, \tag{4.2.5}$$

where $A(\xi^*)$ is a displacement function which is, as yet, undetermined. To proceed further we must consider the flow in the lower deck.

(ii) *The lower deck*

As we have already noted the thickness of the lower deck is $O(R^{-\frac{5}{8}})$. In addition to (4.2.1) we therefore introduce the new variable ζ_* where

$$\zeta_* = R^{-\frac{1}{8}}\zeta, \tag{4.2.6}$$

and in order effect a match between the oncoming flow and the lower deck it is necessary to introduce new variables p_*, ψ_*, s_* as

$$p_* = R^{\frac{1}{4}}(p - p_0), \quad \psi_* = R^{\frac{1}{4}}\psi, \quad s_* = R^{\frac{1}{4}}s. \tag{4.2.7}$$

Substituting (4.2.1), (4.2.6) and (4.2.7) into (3.4) shows that p_*, ψ_* and s_* satisfy

$$\frac{\partial\psi_*}{\partial\zeta_*}\frac{\partial^2\psi_*}{\partial\xi^*\partial\zeta_*} - \frac{\partial\psi_*}{\partial\xi^*}\frac{\partial^2\psi_*}{\partial\zeta_*^2} = -\frac{dp_*}{d\xi^*} + \frac{\partial^3\psi_*}{\partial\zeta_*^3},$$
$$\frac{\partial\psi_*}{\partial\zeta_*}\frac{\partial^2 s_*}{\partial\xi^*\partial\zeta_*} - \frac{\partial\psi_*}{\partial\xi^*}\frac{\partial^2 s_*}{\partial\zeta_*^2} = \frac{\partial^3 s_*}{\partial\zeta_*^3}.$$

The boundary conditions to be satisfied by the solution of these equations at $\zeta_* = 0$ are

$$\psi_* = \frac{\partial\psi_*}{\partial\zeta_*} = s_* = \frac{\partial s_*}{\partial\zeta_*} = 0. \tag{4.2.9}$$

In addition we require that as $\zeta_* \to \infty$

$$\psi_* \sim a_0\zeta_*^2 + \frac{c_0\bar{c}_1\alpha}{2a_0^{\frac{1}{2}}}\zeta_*^{\frac{3}{2}} + 2a_0A(\xi^*)\zeta_*,$$
$$s_* \sim A_0\zeta_*^2 + \frac{A_0c_0\bar{c}_1\beta}{2a_0^{\frac{3}{2}}}\zeta_*^{\frac{3}{2}} + 2A_0A(\xi^*)\zeta_*, \tag{4.2.10}$$

in order to effect a match with the main deck. Also, as $\xi^* \to -\infty$, the lower-deck solution will match the oncoming flow provided that

$$\psi_* \sim a_0^{\frac{1}{3}}\eta^2(-\xi^*)^{\frac{2}{3}} + \frac{c_0\bar{c}_1}{2a_0}(-\xi^*)^{\frac{1}{2}}f(\eta),$$
$$s_* \sim \frac{A_0}{a_0^{\frac{2}{3}}}\eta^2(-\xi^*)^{\frac{2}{3}} + \frac{A_0c_0\bar{c}_1}{2a_0^2}(-\xi^*)^{\frac{1}{2}}g(\eta), \tag{4.2.11}$$

where f, g and η are as in equation (4.1.5).

Now, since in (4.2.10) the displacement function $A(\xi^*)$ is undetermined, or in (4.2.8) the pressure p_* is undetermined, the solution for the lower deck cannot be completed independently of the flow outside the main deck, i.e. in the upper deck. This we now consider.

(iii) *The upper deck*

The outflow from the main deck provides a disturbance which is larger than that associated with classical boundary-layer theory and it is this which is responsible for the triple-deck interaction. The upper deck has length scales $O(R^{-\frac{3}{8}})$ both parallel and normal to the wall and so in this region, in addition to (4.2.1) we introduce a new normal co-ordinate

$$\zeta^* = R^{\frac{3}{8}}(z/\ell). \tag{4.2.12}$$

The outflow from the main deck is given, from (3.2), (3.3), (4.2.2), (4.2.5) and (4.2.12) as

$$V_z = -R^{-\frac{1}{4}}c_0\left(\frac{\ell}{r}\right)^{\frac{1}{2}}A'(\xi^*) - R^{-\frac{3}{8}}\frac{\ell\zeta^*}{r}, \tag{4.2.13}$$

where the term $O(R^{-\frac{3}{8}})$ in (4.2.13) arises, not from viscous effects, but from conical nature of the inviscid flow.

In the upper deck the flow to leading order is inviscid and, since no vorticity is convected into it, irrotational. It is convenient therefore to introduce a velocity potential ϕ as

$$\phi = \frac{r}{\ell} + R^{-\frac{1}{4}}\phi_2 + \ldots\ldots, \tag{4.2.14}$$

where ϕ_2 satisfies $\Delta^2\phi_2 = 0$ or invoking the slender nature of the flow

$$\frac{\partial^2\phi_2}{\partial\eta^{*2}} + \frac{\partial^2\phi_2}{\partial\zeta^{*2}} = 0, \tag{4.2.15}$$

where $\eta^* = r\xi^*/\ell$. The solution of (4.2.15) is required to vanish as $\eta^{*2} + \zeta^{*2} \to \infty$ and, on $\zeta^* = 0$, to satisfy

$$\frac{\partial \phi_2}{\partial \zeta^*} = -c_0\left(\frac{\ell}{r}\right)^{\frac{1}{2}} A'(\ell\eta^*/r), \tag{4.2.16}$$

from (4.2.13). Thus from (4.2.15), (4.2.16) we have

$$\phi_2 = c_0\left(\frac{\ell}{r}\right)^{\frac{1}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A'(\ell\eta_1^*/r)(\eta^* - \eta_1^*)}{(\eta^* - \eta_1^*)^2 + \zeta^{*2}} d\eta_1^*. \tag{4.2.17}$$

Now, with $V_{z2} = \partial\phi_2/\partial\zeta^*$ and, from the Euler equations, $c_0 \partial V_{z2}/\partial\eta^* = -\partial p_2/\partial\zeta^*$, we may calculate the perturbation pressure p_2 in the upper deck. In particular on $\zeta^* = 0$ we have

$$p_2(\xi^*) = \frac{c_0^2}{\pi}\left(\frac{\ell}{r}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{A'(\xi_1^*)d\xi_1^*}{\xi^* - \xi_1^*}, \tag{4.2.18}$$

which, since the pressure is constant across the main deck, provides an expression for the pressure when matched with the lower deck. A numerical solution for the fundamental problem associated with the lower deck, that is $(4.2.8)_1$, (4.2.18) with (4.2.9), $(4.2.10)_1$, $(4.2.11)_1$ has been presented by Smith [7]. In this solution the flow separates smoothly from the surface in contrast to the singular behaviour [10] which is typical of boundary-layer separation in a prescribed pressure gradient.

The final stage of the analysis is to examine the flow beyond the triple-deck interaction region, to ensure that our solution is consistent with that in the post-interaction region.

5. The post-interaction region

Downstream from the interaction region, where separation has taken place, the dominant feature of the flow is the shear layer centred upon the vortex sheet of the inviscid flow. This shear layer separates two regions in which velocities are $O(1)$ and in that respect differs from its counterpart in two dimensions which separates a region in which velocities are $O(1)$ from the 'dead-air' region associated with the separation bubble. There will, in addition, be a boundary layer at the solid surface in which the inviscid velocity (2.3) is adjusted to zero. Both shear layer and boundary layer are discussed on the scale $\xi = O(1)$ in Section 5.1 below. Neither of these regions match upstream with the triple-deck region and we argue in Section 5.2 that there is an adjustment region in which $\xi = O(R^{-\frac{21}{88}})$ between the triple-deck and the emerging shear layer and boundary layer. The situation is shown schematically in Fig. 2.

5.1 The shear and boundary layers

In the shear layer, which lies close to the boundary just downstream from separation as may be seen from equation (2.1), we define co-ordinates ξ_s, ζ_s where the former measures distance along the sheet and the latter is scaled as in (3.1) and is nominally centred upon the vortex sheet. Note that to first order ξ and ξ_s are indistinguishable, and that the inviscid velocities V_r, V_ξ and pressure gradient $\partial p/\partial \xi$ between the vortex sheet and boundary are indistinguishable from their surface values (2.3) when $|\xi| \ll 1$. In the separated part of the boundary layer we can assume that the main-deck profile is the dominant feature, so that from (4.1.3)

$$\begin{aligned} \psi &\sim a_0 \zeta_s^2 + a_1 \zeta_s^3 + a_3 \zeta_s^5 + \ldots\ldots, \\ s &\sim A_0 \zeta_s^2 + A_3 \zeta_s^5 + \ldots\ldots . \end{aligned} \tag{5.1.1}$$

In the shear layer we develop the solution of (3.4) as

$$\begin{aligned} \psi &= \xi_s^{\frac{1}{3}} f_1(\eta_s) + \xi_s f_2(\eta_s) + \xi_s^{\frac{4}{3}} f_3(\eta_s) + \ldots\ldots, \\ s &= \xi_s^{\frac{1}{3}} g_0(\eta_s) + \xi_s^{\frac{2}{3}} g_1(\eta_s) + \xi_s g_2(\eta_s) + \ldots\ldots, \end{aligned} \tag{5.1.2}$$

where

$$\eta_s = \zeta_s / \xi_s^{\frac{1}{3}} .$$

In the expansions (5.1.2) the role of the various terms is readily identified. Thus g_0 is required to match with the radial velocity $(2.2)_1$ and so

$$g_0'(\infty) = 0, \quad g_0'(-\infty) = 1. \tag{5.1.3}$$

The next term in the series for s is required to match with the separated main deck so that, from (5.1.1), (5.1.2), (2.3)

$$g_1 \sim A_0 \eta_s^2 \text{ as } \eta_s \to \infty, \quad g_1'(-\infty) = 0.$$

In $(5.1.2)_1$ the leading term matches with the main-deck solution (5.1.1), so that

$$f_1 \sim a_0 \eta_s^2 \text{ as } \eta_s \to \infty, \quad f_1'(-\infty) = 0, \tag{5.1.4}$$

as does f_2. The next term involving f_3 is required to match with the inviscid velocity V_ξ in (2.3) so that

$$f_3'(\infty) = 0, \quad f_3'(-\infty) = \frac{4}{2M+5} . \tag{5.1.5}$$

All the functions f_i, g_i satisfy ordinary differential equations of which the first pair are

$$\begin{aligned} f_1''' + \tfrac{2}{3} f_1 f_1'' - \tfrac{1}{3} f_1'^2 &= 0, \\ g_0''' + \tfrac{2}{3} f_1 g_0'' &= 0, \end{aligned}$$

with

$$f_1 \sim a_0\eta^2 \text{ as } \eta \to \infty, \quad g_0'(\infty) = 0,$$
$$f_1(0) = g_0(0) = 0, \quad f_1'(-\infty) = 0, \quad g_0'(-\infty) = 1. \tag{5.1.6}$$

Also in this region, where $\xi = O(1)$ with $|\xi| \ll 1$, a boundary layer will be established on the surface $z = 0$ appropriate to the inviscid surface velocity $(2.3)_{1,2}$. As Smith [1] observes, with the small favourable pressure gradient $(2.3)_3$ and a slowly diverging external flow there is no tendency for the boundary layer in this region to separate. We note that with the surface velocities as in (2.3) the boundary-layer equations (3.4) admit a separable solution. However we report that the resulting ordinary differential equations have no solution for $M \lesssim 2.04$. We take this to mean that for $M = 0, 1, 2$ the boundary layer is not determined by local conditions but depends upon the flow conditions away from the separation region under discussion.

The shear-layer and boundary-layer solutions discussed in this section will not match satisfactorily with the triple-deck solution of Section 4. This is most readily seen from a further discussion of the shear-layer solution (5.1.2). Thus if we introduce the lower-deck length scales (4.2.1) and (4.2.6) into $(5.1.2)_1$ then we see that $\psi = O(R^{-\frac{1}{4}})$ as we require (see (4.2.7)) for the lower deck. However introducing these same scales into $(5.1.2)_2$ gives $s = O(R^{-\frac{1}{8}})$, not $O(R^{-\frac{1}{4}})$ as is required by (4.2.7) if a match with the triple-deck is to be effective. The reason for this mis-match is associated with the leading term of the expansion for s in $(5.1.2)_2$. Thus, the analysis of Section 4.2 shows that the relatively small lower-deck velocity components parallel to the surface are $O(R^{-\frac{1}{8}})$ whereas the leading term in the expansion for s in $(5.1.2)_2$ is specifically constructed to match an $O(1)$ radial velocity as in (5.1.3).

It is clear that some transition region is needed if a match with the triple-deck region is to be effected. This is the adjustment region labelled 3(iii) in Fig. 2 and is discussed below.

5.2. *The adjustment region*

To determine the scale of this adjustment region we first examine the viscous entrainment into the shear layer centred upon the vortex sheet of the inviscid flow. From a knowledge of this entrainment velocity we can then determine the magnitude of the correction to the inviscid velocities in (2.3) due to viscous effects. Thus, from $(3.3)_2$ and (5.1.2) we deduce that

$$V_z \sim \tfrac{1}{2}\zeta_s g_0' - \tfrac{3}{2}\xi_s^{\frac{1}{3}} g_0 - \xi_s^{-\frac{1}{3}}\left(\tfrac{2}{3}f_1 - \tfrac{1}{3}\eta_s f_1'\right) \text{ as } \xi_s \to 0$$
$$\sim -\zeta_s - \tfrac{2}{3}C\xi_s^{-\frac{1}{3}} \text{ as } \zeta_s \to -\infty, \text{ where } C = f_1(-\infty). \tag{5.2.1}$$

The leading term of (5.2.1) simply accommodates the expanding inviscid conical flow, the second term is a measure of the viscous entrainment and, when (3.2) is used, shows that the actual entrainment velocity is

$$O(R^{-\frac{1}{2}}\xi^{-\frac{1}{3}}). \tag{5.2.2}$$

To assess the effect of this upon the other velocity components consider a control surface with the vortex sheet as one side, the plane $z = 0$ as one side with the remaining sides, the plane $\xi = \xi_0$ where $1 \gg \xi_0 > 0$ and the surface r, $r + \Delta r = \text{const}$. From (2.1) the height of this region is $O(R^{-\frac{1}{16}}\xi^{\frac{3}{2}})$. Now consider the volume balance within this control surface. Into the vortex sheet volume is entrained, from (5.2.2), at the rate

$$O(R^{-\frac{1}{2}}\xi^{\frac{2}{3}}\,\Delta r). \tag{5.2.3}$$

If the perturbation velocities parallel to the surface are ΔV_r and ΔV_ξ respectively then the other contributions to this perturbation volume balance are

$$O(R^{-\frac{1}{16}}\xi^{\frac{5}{2}}\Delta r \Delta V_r), \quad O(R^{-\frac{1}{16}}\xi^{\frac{3}{2}}\Delta r \Delta V_\xi), \tag{5.2.4}$$

respectively. Thus, for volume conservation we require

$$\Delta V_r = O(R^{-\frac{7}{16}}\xi^{-\frac{11}{6}}), \quad \Delta V_\xi = O(R^{-\frac{7}{16}}\xi^{-\frac{5}{6}}). \tag{5.2.5}$$

From (2.3) we may now write, incorporating these perturbations

$$\begin{aligned} V_r &= 1 + \Delta V_r + O(\xi^2), \\ V_\xi &= -\frac{4}{2M+5}\xi + \Delta V_\xi + O(\xi^2), \end{aligned} \tag{5.2.6}$$

and we see from (5.2.5), (5.2.6) that as $\xi \to 0$ the supposed perturbation is comparable with the 'main' flow when

$$\xi = O(R^{-\frac{21}{88}}). \tag{5.2.7}$$

This, when combined with (2.1) and (3.1) to give

$$\zeta = O(R^{\frac{7}{88}}), \tag{5.2.8}$$

provides the scale for the adjustment region. Also, from (3.3) and (5.2.6) we have

$$\psi = O(R^{-\frac{14}{88}}), \quad s = O(R^{\frac{7}{88}}). \tag{5.2.9}$$

If we adopt the scales (5.2.7) to (5.2.9) for the adjustment region and allow a subscript 'a' to denote a variable appropriate to that region then viscous terms are negligible and from (3.4) we have

$$\begin{aligned} &-\left(\frac{3}{2}s_a + \frac{\partial\psi_a}{\partial\xi_a}\right)\frac{\partial^2 s_a}{\partial\zeta_a^2} + \frac{\partial\psi_a}{\partial\zeta_a}\,\frac{\partial^2\zeta_a}{\partial\xi_a\partial\zeta_a} = 0, \\ &-\left(\frac{3}{2}s_a + \frac{\partial\psi_a}{\partial\xi_a}\right)\frac{\partial^2\psi_a}{\partial\zeta_a^2} + \frac{\partial\psi_a}{\partial\zeta_a}\,\frac{\partial^2\psi_a}{\partial\xi_a\partial\zeta_a} + \frac{\partial\psi_a}{\partial\zeta_a}\,\frac{\partial s_a}{\partial\zeta_a} + \frac{dp_a}{d\xi_a} = 0. \end{aligned} \tag{5.2.10}$$

The boundary conditions require

$$\psi_a(\xi_a, 0) = s_a(\xi_a, 0) = 0, \tag{5.2.11}$$

and also, as $\xi_a \to \infty$,

$$\begin{aligned} \frac{\partial \psi_a}{\partial \zeta_a} &\sim -\frac{4}{2M+5}\,\xi_a + O(\xi_a^{-\frac{5}{6}}), \\ \frac{\partial s_a}{\partial \zeta_a} &\sim 1 + O(\xi_a^{-\frac{11}{6}}), \end{aligned} \tag{5.2.12}$$

in order to effect a match with (5.2.6). It will also be necessary for the solution in this region to match, on

$$\zeta_a = \mu_a\, \xi_a^{\frac{3}{2}}, \tag{5.2.13}$$

where $\mu_a = R^{\frac{1}{16}}\mu$ in (2.1), with the solution in the shear layer which separates the adjustment region from the separating main deck. In this shear layer, whose continuance has been discussed in (5.1.2), we have

$$\psi_a = \xi_{sa}^{\frac{2}{3}}\, f_1(\eta_{sa}),\ \ \eta_{sa} = \zeta_{sa}/\xi_{sa}^{\frac{1}{3}},\ \ \zeta_{sa} = R^{\frac{7}{88}}\zeta_s, \tag{5.2.14}$$

and f_1 as in $(5.1.2)_1$. Further if in this shear layer we write $s = R^{-\frac{7}{88}}\bar{s}$ then, as $\xi_{sa} \to \infty$ we have from $(5.1.2)_2$

$$\bar{s} \sim \xi_{sa}^{\frac{1}{3}}\, g_0(\eta_{sa}) + R^{-\frac{7}{88}}\xi_{sa}^{\frac{2}{3}}\, g_1(\eta_{sa}) + \ldots\ldots, \tag{5.2.15}$$

where g_0, g_1 are as in $(5.1.2)_2$. This leads us to write, for the shear layer associated with the adjustment region

$$\bar{s} = s_0(\xi_{sa}, \zeta_{sa}) + O(R^{-\frac{7}{88}})$$

where s_0 satisfies

$$-\frac{\partial \psi_a}{\partial \zeta_{sa}}\frac{\partial^2 s_0}{\partial \zeta_{sa}^2} + \frac{\partial \psi_a}{\partial \zeta_{sa}}\frac{\partial^2 s_0}{\partial \xi_{sa}\partial \zeta_{sa}} = \frac{\partial^3 s_0}{\partial \zeta_{sa}^3}, \tag{5.2.16}$$

from $(3.4)_1$. The solution of (5.2.16) is required to match with the adjustment region as $\zeta_{sa} \to -\infty$ and also $\partial s_0/\partial \zeta_{sa} \to 0$ as $\zeta_{sa} \to \infty$.

It now remains to consider how this solution for the adjustment region matches with the lower deck in the triple-deck interaction region. The situation is little different at this point from the two-dimensional case considered by Sychev [2], and so we require as $\xi_a \to 0$

$$\psi_a \sim \xi_a^{\frac{2}{3}} \chi(\eta_a), \quad s_a \sim \xi_a^{\frac{29}{12}} \phi(\eta_a), \quad p_a \sim \xi_a^{-\frac{5}{3}} \pi_0,$$
$$\eta_a = \zeta_a / \xi_a^{\frac{3}{2}}. \tag{5.2.17}$$

The forms of ψ_a, s_a in $(5.2.17)_{1,2}$ ensure that in the triple-deck region, where $\xi = O(R^{-\frac{3}{8}})$, ψ and s are both $O(R^{-\frac{1}{4}})$ as required in the lower deck where $\zeta = O(R^{-\frac{1}{8}})$. The functions $\chi(\eta_a)$, $\phi(\eta_a)$ satisfy

$$2\chi\chi'' + \tfrac{5}{2}\chi'^2 + 5\pi_0 = 0,$$
$$-\tfrac{3}{2}\chi\phi'' + \tfrac{11}{2}\chi'\phi' = 0, \tag{5.2.18}$$

where

$$\chi(0) = \phi(0) = 0,$$
$$\chi(\mu_a) = C, \quad \phi(\mu_a) = D \tag{5.2.19}$$

where $C = f_1(-\infty)$ and D is a constant which will be determined from the solution of (5.2.16) as $\xi_{sa} \to 0$. The form of ψ in $(5.2.17)_1$ is the same as that adopted by Sychev [2] in the two-dimensional case and has been shown by Smith [7], in his numerical calculations, to be appropriate.

This reconciliation of the adjustment region with the lower deck of the triple-deck interaction region completes a self-consistent description of the post-interaction flow.

6. Conclusions

In the preceding three sections we have shown how the inviscid solution, presented by Smith [1], for the flow in the neighbourhood of a separation line on a smooth surface in a slender conical flow can be embedded within a self-consistent high Reynolds number theory which itself is dependent upon the interaction ideas associated with triple-deck theory. In the proposed slender, conical flow, with rapid changes perpendicular to the separation line predicted by the inviscid analysis, it is not surprising that the essential features of the high Reynolds number flow which emerge are closely related to those proposed for the two-dimensional case by Sychev [2]. The main difference between the present work and the two-dimensional theory is in the region downstream from separation. In the present case the open type of separation is such that the line of separation divides the flow into regions in which the flow velocities are $O(1)$. By contrast in two-dimensional flow the separation line divides the flow into a region where velocities are $O(1)$ and, in the 'dead-air' region associated with the separation bubble, a region in which velocities are very much smaller. Thus in the present case the flow structure immediately downstream from separation is more complicated than in two-dimensional flow.

REFERENCES

[1] J. H. B. Smith, *Behaviour of a vortex sheet separating from a smooth surface*, RAE TR 77058 (1977), 62 pp.
[2] V. V. Sychev, On laminar separation, *Mekhanika Zhidosti i Gaza* 3 (1972) 47-59.
[3] J. H. B. Smith, Improved calculations of leading-edge separation from slender, thin, delta wings, *Proc. R. Soc.* A306 (1968) 67-90.
[4] J. Nutter, Ph. D. Thesis, University of East Anglia (1978).
[5] B. Thwaites, *Incompressible Aerodynamics*, Clarendon Press, Oxford, England (1960) 112-175.
[6] K. Stewartson, Multistructured boundary layers on flat plates and related bodies, *Adv. appl. Mech.* 14 (1974) 145-239.
[7] F. T. Smith, The laminar separation of an incompressible fluid streaming past a smooth surface, *Proc. R. Soc.* A356 (1977) 443-463.
[8] F. T. Smith, Private communication.
[9] S. N. Brown, Singularities associated with separating boundary layers, *Phil. Trans. R. Soc.* A257 (1965) 409-444.
[10] S. Goldstein, On laminar boundary-layer flow near a position of separation, *Q. Jl Mech. appl. Math.* 1 (1948) 43-69.
[11] K. Stewartson, On Goldstein's theory of laminar separation, *Q. Jl Mech. appl. Math.* 11 (1958) 399-410.
[12] A. F. Messiter, Boundary-layer flow near the trailing edge of a flat plate, *SIAM J. appl. Math.* 18 (1970) 241-257.
[13] K. Stewartson, On the flow near the trailing edge of a flat plate II, *Mathematika* 16 (1969) 106-121.